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Wild automorphisms of generic matrix algebras

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Abstract

Bergman showed that the algebra of two 2×2 generic matrices has a wild automorphism, using a central polynomial. We construct new wild automorphisms of the algebra of two $n \times n$ generic matrices, $n \geq 3$, without using central polynomials. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let k be a field. Let $k\langle \mathscr{X}, \mathscr{Y} \rangle$ be the free associative algebra generated by non-commutative variables $\mathscr{X}, \mathscr{Y}$ over k and let $k[x, y]$ be the commutative polynomial ring. Let $k\{X, Y\}_n$ be a generic matrix algebra generated by two $n \times n$ generic matrices,

$$X = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & \dots & y_{1n} \\ \vdots & \vdots & \vdots \\ y_{n1} & \dots & y_{nn} \end{pmatrix},$$

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where x_{ij}, y_{ij} are commutative variables for $1 \leq i, j \leq n$. For a k-algebra R, $\mathrm{Aut}(R)$ denotes the group of k-algebra automorphisms of R. In particular, the homomorphisms $\phi, \psi : k\langle \mathscr{X}, \mathscr{Y} \rangle \longrightarrow k\langle \mathscr{X}, \mathscr{Y} \rangle$ defined by $\phi(\mathscr{X}) = a\mathscr{X} + b\mathscr{Y}$, $\phi(\mathscr{Y}) = c\mathscr{X} + d\mathscr{Y}$, and $\psi(\mathscr{X}) = \mathscr{X} + g(\mathscr{Y})$, $\psi(\mathscr{Y}) = \mathscr{Y}$, where $a, b, c, d \in k$, $ad - bc \neq 0$, and $g(\mathscr{Y})$ is a polynomial in $\mathscr{Y}$ with coefficients in k, are called *elementary* automorphisms. An automorphism which lies in the group generated by the elementary automorphisms is called *tame* and an automorphism which is not tame is called *wild*. The following theorem was proved by Jung [3] and van der Kulk [6].

Theorem 1.1. *Every automorphism in* $\mathrm{Aut}(k[x, y])$ *is tame.*

For the group $\mathrm{Aut}(k\langle \mathscr{X}, \mathscr{Y} \rangle)$, Makar–Limanov [4] and Czerniakiewicz [2] proved,

Theorem 1.2. *The natural homomorphism* $\tau^* : \mathrm{Aut}(k\langle \mathscr{X}, \mathscr{Y} \rangle) \longrightarrow \mathrm{Aut}(k[x, y])$ *is an isomorphism, in other words, every automorphism of* $k\langle \mathscr{X}, \mathscr{Y} \rangle$ *is tame.*

But this is not the case for $k\{X, Y\}_2$. Bergman [1] showed

Theorem 1.3. *The automorphisms* $\alpha, \beta : k\{X, Y\}_2 \longrightarrow k\{X, Y\}_2$, *defined by* $\alpha(X) = X + [X, Y]^2$, $\alpha(Y) = Y$ *and* $\beta(X) = X + Y[X, Y]^2$, $\beta(Y) = Y$, *where* $[X, Y]$ *is the commutator of* X *and* Y, *are wild.*

The key point is that $[X, Y]^2$ is a central element of $k\{X, Y\}_2$ so that $\alpha^{-1}(X) = X - [X, Y]^2$, $\beta^{-1}(X) = X - Y[X, Y]^2$ and both α^{-1} and β^{-1} fix Y. In the next section, we shall show that there is a wild automorphism whose images include non-central elements in $k\{X, Y\}_n$ for $n \geq 3$.

2. New automorphisms

Let $x_1, \ldots, x_{n+1}$ be commutative variables. For each polynomial $\sum c_P\, x_1^{p_1} \cdots x_{n+1}^{p_{n+1}}$ in $k[x_1, \ldots, x_{n+1}]$, one associates a non-commutative polynomial $\sum c_P\, \mathscr{X}^{p_1}\, \mathscr{Y}_1\, \mathscr{X}^{p_2}\, \mathscr{Y}_2 \cdots \mathscr{X}^{p_n}\, \mathscr{Y}_n\, \mathscr{X}^{p_{n+1}}$. Let $\mathscr{F}(\mathscr{X}, \mathscr{Y}_1, \ldots, \mathscr{Y}_n)$ be the non-commutative polynomial corresponding to

$$f(x_1, \ldots, x_{n+1}) = \prod_{i=2}^{n} (x_1 - x_i)(x_{n+1} - x_i) \prod_{2 \leq j < k \leq n} (x_j - x_k) = \sum c_P x_1^{p_1} \cdots x_{n+1}^{p_{n+1}}.$$

Theorem 2.1. *If* $\Phi : k\{X, Y\}_n \longrightarrow k\{X, Y\}_n$ *is a map under which the images of the generators are* $\Phi(X) = X$, $\Phi(Y) = Y + F(X, Y)$ *where* $n \geq 3$ *and* $F(X, Y) = \mathscr{F}(X, Y, \ldots, Y)$, *then* Φ *is a wild automorphism of* $k\{X, Y\}_n$.

Proof. First of all, we show that X and $F(X, Y)$ commute. We may assume X is a diagonal matrix (x_{ii}) and $Y = \left(y_{ij}\right) = \sum y_{ij} e_{ij}$ where e_{ij}'s are matrix units [5, Proposiition 1.3.13, p. 17]. Then

$$X^{p_1} y_{i_1j_1} e_{i_1j_1} X^{p_2} y_{i_2j_2} e_{i_2j_2} \cdots X^{p_n} y_{i_nj_n} e_{i_nj_n} X^{p_{n+1}}$$
$$= x_{i_1i_1}^{p_1} \cdots x_{i_ni_n}^{p_n} x_{j_nj_n}^{p_{n+1}} y_{i_1j_1} \cdots y_{i_nj_n} e_{i_1j_1} \cdots e_{i_nj_n}$$

and

$$\mathscr{F}(X, y_{i_1j_1} e_{i_1j_1}, y_{i_2j_2} e_{i_2j_2}, \dots, y_{i_nj_n} e_{i_nj_n})$$
$$= f(x_{i_1i_1}, \dots, x_{i_ni_n}, x_{j_nj_n}) y_{i_1j_1} \cdots y_{i_nj_n} e_{i_1j_1} \cdots e_{i_nj_n}.$$

The polynomial $f(x_{i_1i_1},\dots,x_{i_ni_n},x_{j_nj_n})$ vanishes unless $\{i_1,\dots,i_n\}$ is a permutation of $\{1,\dots,n\}$ and $j_n = i_1$. And $e_{i_1j_1} \dots e_{i_nj_n} = 0$ unless $j_1 = i_2$, $j_2 = i_3$, …, $j_{n-1} = i_n$. When $\{i_1,\dots,i_n\}$ is a permutation of $\{1,\dots,n\}$ and $j_1 = i_2$, $j_2 = i_3$, …, $j_{n-1} = i_n$, $j_n = i_1$, both $f(x_{i_1i_1},\dots,x_{i_ni_n},x_{j_nj_n}) \neq 0$ and $e_{i_1j_1} \dots e_{i_nj_n} = e_{i_1i_1}$, i.e. $F(X,Y) = \mathscr{F}(X,Y,\dots,Y)$ is a diagonal matrix. Thus X and $F(X,Y)$ commute. Furthermore, $f(x_{i_1i_1},\dots,x_{i_ni_n},x_{j_nj_n})$ is of degree $2(n-1)$ for $x_{i_1i_1}$ and of degree n for $x_{i_2i_2}$, …, $x_{i_ni_n}$ if $\{i_1,\dots,i_n\}$ is a permutation of $\{1,\dots,n\}$ and $j_1 = i_2$, $j_2 = i_3$, …, $j_{n-1} = i_n$, $j_n = i_1$. This fact implies that $F(X,Y)$ is not a scalar matrix, i.e. $F(X,Y)$ is not a central element.

Now we need to show $F(X,F(X,Y)) = 0$ so that the automorphism $\Theta(X) = X$, $\Theta(Y) = Y - F(X,Y)$ is the inverse of Φ. If $F(X,Y) = \sum c_P X^{p_1} Y \cdots X^{p_n} Y X^{p_{n+1}}$, then

$$\begin{aligned} F(X,F(X,Y)) &= \sum c_P X^{p_1} F(X,Y) \cdots X^{p_n} F(X,Y) X^{p_{n+1}} \\ &= \sum c_P X^{p_1+\cdots+p_{n+1}} F(X,Y)^n = \left(\sum c_P X^{p_1+\cdots+p_{n+1}}\right) F(X,Y)^n. \end{aligned}$$

But $\sum c_P X^{p_1+\cdots+p_{n+1}} = 0$.

To prove that Φ is wild, let us consider the following diagram:

$$\begin{tikzcd} k\langle \mathscr{X}, \mathscr{Y} \rangle \arrow[r, "\mu"] \arrow[dr, "\tau"'] & k\{X,Y\}_n \arrow[d, "\nu"] \\ & k[x,y] \end{tikzcd}$$

where μ, ν, τ are k-algebra homomorphisms, the kernel of μ is the ideal of polynomial identities of $n \times n$ matrices in two variables, and the kernel of ν is the commutator ideal. These homomorphisms induce the commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(k\langle \mathscr{X}, \mathscr{Y} \rangle) \arrow[r, "\mu^*"] \arrow[dr, "\tau^*"'] & \mathrm{Aut}(k\{X,Y\}_n) \arrow[d, "\nu^*"] \\ & \mathrm{Aut}(k[x,y]) \end{tikzcd}$$

Since τ^* is an isomorphism by Theorem 1.2, μ^* is a monomorphism and ν^* is an epimorphism. An automorphism in Aut $(k\{X,Y\}_n)$ is tame if it is the image of μ^*.

Clearly $\Phi \neq \mathrm{id}$ but $v^*(\Phi) = \mathrm{id}$. Hence, the fact that Φ is not in the image of μ^* implies that the automorphism Φ is wild. $\square$

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